A new rank parity computing machine

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Main results

Theorem (Constantinou, Dokchitser, Green, Morgan)

Assume $\# X$ is finite. For all smooth, projective curves over number fields X/K

$$\operatorname{rank}(\mathrm{Jac}_X/K) \equiv \sum_v \lambda(X/K_v) \mod 2$$

where $\lambda \in \{0, 1\}$ is an explicit invariant computed from curves over local fields.

Work in progress theorem (Dokchitser, Green, Morgan)

Assume $\# X$ is finite. The Birch and Swinnerton-Dyer conjecture correctly predicts the parity of $\operatorname{rank}(\mathrm{Jac}_X/K)$ for all nice hyperelliptic curves over number fields X/K.

Theorem (Green, Maistret: $p = 2$ and E has CM)

The p-parity conjecture holds for elliptic curves over totally real number fields.

BSD and the parity conjecture

Birch and Swinnerton-Dyer conjecture

$$\mathrm{rank}(\mathrm{Jac}_X) = \mathrm{ord}_{s=1} L(\mathrm{Jac}_X; s)$$

+

Conjectural functional equation

$$L\ (\mathrm{Jac}_X; s) = w(\mathrm{Jac}_X) L\ (\mathrm{Jac}_X; 2 \quad s)$$

The Parity Conjecture

Let K be a number eld and $X=K$ a curve. Then

$$(\quad 1)^{\mathrm{rank}(\mathrm{Jac}_X = \quad)} \qquad K$$

Applications of local formulae

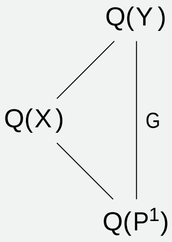
Let E/K be a semistable elliptic curve. Assuming BSD, or finiteness of Ш,

$$\mathrm{rank}(E/K) \equiv \#\{v \mid \infty\} + \#\{v \nmid \infty,\ E/K_v \text{ split multiplicative}\} \mod 2.$$

- $E/\mathbb{Q}$

Ingredient 1 for the parity computing machine: eld diagrams

Let $X=Q$ be a curve and $: X \;!\; P^1$.

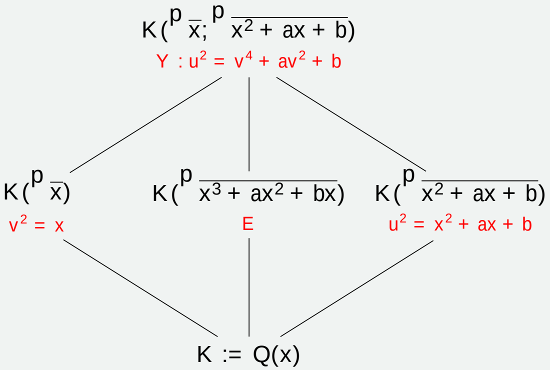


- $Q(Y)^H = Q(Y=H)$
- $(Jac_Y(Q) \quad Q)^H = Jac_{Y=H}(Q) \quad Q$
- Tate modules
- Selmer groups
- Height pairings

Example

Let $E : y^2 = x^3 + ax^2 + bx$ and $: E \;!\; P^1$;

$$(x;y) \;7!\; x:$$



Ingredient 2 for the parity computing machine: Brauer relations

Let G be a nite group.

The parity computing machine

Theorem (Constantinou, Dokchitser, Green, Morgan)

Let $Y/\mathbb{Q}$ be smooth, projective such that $\#\text{Ш}_{\mathrm{Jac}_Y}[\ell^\infty]$ is finite. Assume $Y \to \mathbb{P}^1$ is a Galois cover and let $\Theta = \sum_i H_i - \sum_j H'_j$ be a Brauer relation for its Galois group. Then

$$\mathrm{ord}_\ell \left(\frac{\prod_i \mathrm{Reg}_{\mathrm{Jac}_{Y/H_i}}}{\prod_j \mathrm{Reg}_{\mathrm{Jac}_{Y/H'_j}}} \right) \equiv \sum_v \Lambda_\Theta(Y/\mathbb{Q}_v) \mod 2$$

where Λ_Θ is an expression in Θ and local data for $Y/\mathbb{Q}_v$.

Example

The parity computing machine

We recover local formulae for:

- E admitting a cyclic ℓ-isogeny (Cassels), if $E(K)[\ell] \neq \{O\}$ then $D_{2\ell}$
- Jac_X for X hyperelliptic over quadratic extensions (Kramer, Tunnell; Morgan), $C_2 \quad C_2$
- Jac_X for X of genus 2 with a Richelot isogeny (Dokchitser, Maistret), D_8
- Jac_X for X of genus 3 such that G acts on $\mathrm{Jac}_X[2]$ by a K 2-group (Docking). S_4

Theorem (Constantinou, Dokchitser, Green, Morgan)

Thank you for your attention!